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Institute of Mathematical Sciences
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The Behavior of the Solutions of $\Delta U = F(x,U)$ in the Neighborhood of a Point

CLAUS MÜLLER

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THE BEHAVIOR OF THE SOLUTIONS OF $\Delta U = F(x, U)$ IN THE NEIGHBORHOOD OF A POINT

by

Claus Müller

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Morris Kline

Morris Kline
Project Director

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ABSTRACT

The solutions of the equation $\Delta u + k^2 U = o$, where the function k^2 is bounded and not necessarily analytic and Δ is the p-dimensional Laplacian, is shown to vanish identically if it vanishes at a point more strongly than any power of the distance from that point. An analogous theorem is proved for the equation $\Delta u = F(x,u)$, where F satisfies a Lipschitz condition.

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I. Introduction

The purpose of this paper is to investigate the local behavior of the solution of the differential equation

(1.1) $$\Delta U = F(x,U),$$

and in particular the behavior of the solutions of

(1.2) $$\Delta U + k^2(x)U = 0$$

in p dimensions. Δ is the p-dimensional Laplacian.

Let (x) stand for the spatial vector in p Euclidean dimensions and denote by $|x|$ the length of that vector. We shall use the notation $U \varepsilon C$, $U \varepsilon C'$, and $U \varepsilon C''$ to indicate respectively, that a function U is continuous, has a continuous derivative, or has a continuous second derivative. The main result of this paper may be formulated as

Theorem I. Let U be in the class C'' for $|x| \leq \alpha$. Further, let U(x) satisfy the inequality

(1.3) $$\int_{\Omega_R} |\Delta U|^2 ds \leq c \int_{\Omega_R} |U|^2 ds$$

uniformly in $|x|$ for $0 < R \leq \alpha$; here Ω_R is the p-dimensional sphere $|x| = R$. Now, if

$$\int_{\Omega_R} |U|^2 ds = o(R^n)$$

for all n as $R \to 0$, then the function U vanishes identically.

Since the assumptions of this theorem are obviously satisfied by the solutions of (2), we have

Theorem II. Let U be a solution of

$$\Delta U + k^2(x)U = 0$$

which is in C'' for $|x| \leq \alpha$. Let $k^2(x)$ be uniformly bounded. Now, if

$$\int_{\Omega_R} |U|^2 ds = o(R^n)$$

for all n as $R \to 0$, then U vanishes identically.

Thus in this paper it is proved that any solution of (2) that vanishes more strongly than any power vanishes identically. Until now this result has been proved in p dimensions only with the assumption that $k^2(x)$ be analytic in (x). A result similar to Theorem II was proved by T. Carleman[1] for two dimensions, but his method cannot be extended to higher dimensions. It seems therefore that the method developed below is the first to give results on the local behavior of the solutions of a general class of nonanalytic differential equations in any number of variables.

One of the characteristic properties of second-order differential equations of elliptic type seems to be that any solution of such an equation is fully determined when its local behavior is known. From Theorem II we see that this is true of the nonanalytic differential equation (2). It is also true of the more general equation (1) if we assume that F(x,U) satisfies a Lipschitz condition.

We say F(x,T) satisfies a Lipschitz condition in the neighborhood of (x_0) and T_0 if there exist positive numbers L, c, and d such that for all (x) in the interval $|x - x_0| \leq c$ and all T_1 and T_2 in the interval $|T - T_0| \leq d$

$$|F(x,T_1) - F(x, T_2)| \leq L\, |T_1 - T_2|. \tag{1.4}$$

We can then prove

Theorem III. Let U_1 and U_2 be two solutions of

$$\triangle U = F(x,U)$$

which are in C'' for $|x - x_o| \leq \alpha$, and let $U_1(x_o) = U_2(x_o)$. Furthermore, let $F(x,T)$ satisfy a Lipschitz condition in the neighborhood of (x_o) and $U_1(x_o)$. Now, if

$$\int_{|x-x_o|=R} |U_1(x) - U_2(x)|^2 ds = o(R^n)$$

for all n as $R \to 0$, then it follows that $U_1(x) = U_2(x)$ identically.

Since the inequality

$$|\triangle U_1 - \triangle U_2| = |F(x, U_1) - F(x,U_2)| \leq L\, |U_1 - U_2| \tag{1.5}$$

holds, for suitable $\alpha' \leq \alpha$, for all (x) in $|x - x_o| \leq \alpha'$, Theorem III is an immediate consequence of Theorem I.

These results may be regarded as an extension of our earlier result for the solutions of (2) in three variables.[2]

We shall prove Theorem I in two steps, stated below as Lemmas 1 and 2. The proofs of these lemmas are given in the later sections. Let us denote by (ξ) the unit vector in the direction of (x), so that

$$(x) = |x|\, (\xi). \tag{1.6}$$

Let $S_{n,j}(\xi)$ be a complete set of orthonormal spherical harmonics of order n in p dimensions. The proof is based on a discussion of the functions

$$F_{n,j}(R) = \int_{\Omega_1} U(R\xi)\, S_{n,j}(\xi)\, ds \tag{1.7}$$

for which second-order ordinary differential equations are derived.

Now we state

Lemma 1. Let $U \varepsilon C''$ for all (x) with $|x| \neq 1$ and let the functions

$$\int_{\Omega_R} |U|^2 ds \text{ and } \int_{\Omega_R} |\Delta U|^2 ds$$

be piecewise continuous as functions of R and satisfy the inequality

$$\int_{\Omega_R} |\Delta U|^2 ds \leq c \int_{\Omega_R} |U|^2 ds$$

uniformly for all R. Suppose that for every spherical harmonic $S_n(\xi)$ the integral

$$\int_{\Omega_1} U(R\xi)\, S_n(\xi) ds$$

considered as a function of R is in C'. Now, if

$$\int_{\Omega_R} |U|^2 ds = \begin{cases} o(R^n) & \text{for } R \to 0 \text{ for all } n \\ O(R^{\beta+p-1}) & \text{for } R \to \infty \text{ and some } \beta, \end{cases}$$

then the function $U(x)$ vanishes identically.

In proving this lemma (see Section II) we shall first derive the differential equation for the integrals (6) as functions of R. The expressions obtained involve ΔU. The proof is then completed by an iteration process in which the inequality (3) is used.

We now extend the range of validity of the inequality (3) from functions defined in an α-neighborhood of the origin to functions defined in all space, as required in Lemma 1. This we do in

Lemma 2. Let $f(\xi)$ and $g(\xi)$ be two functions defined on the unit sphere; $f(\xi) \, \varepsilon \, C''$ and is not identically zero, and $g(\xi) \, \varepsilon \, C'$. Then there is a function $U(x) \varepsilon C''$ defined for $|x| > 1$ such that for any spherical harmonic $S_n(\xi)$

$$\lim_{R \to 1+0} \int_{\Omega_1} U(R\xi)\, S_n(\xi)\, d\omega = \int_{\Omega_1} f(\xi)\, S_n(\xi)\, d\omega$$

$$\lim_{R \to 1+0} \frac{d}{dR} \int_{\Omega_1} U(R\xi)\, S_n(\xi)\, d\omega = \int_{\Omega_1} g(\xi)\, S_n(\xi)\, d\omega,$$

while for all $R \geq 1$,

$$\int_{\Omega_R} |\Delta U|^2\, ds \leq c \int_{\Omega_R} |U|^2\, ds,$$

and for $R \to \infty$,

$$\int_{\Omega_R} |U|^2\, ds = O(R^{\beta+2})$$

where c and β are properly chosen constants.

To prove Theorem I from Lemma 1 and Lemma 2, we observe that no restriction is imposed on Theorem I if we assume that $\alpha > 1$ and that

$$\int_{\Omega_1} |U|^2\, ds > 0, \tag{1.8}$$

since a linear transformation of the form

$$(x') = \lambda(x), \tag{1.9}$$

with a constant $\lambda > 0$, will not alter the assumptions of this theorem. By suitable choice of λ we can always find a function $U(x')$ such that (8) is satisfied. From Lemma 2 it then follows that the domain of definition of this function can be extended so as to satisfy the conditions of Lemma 1.

We proceed now with the proofs of the two lemmas.

II. Proof of Lemma 1

Let $S_{n,j}(\xi)$ be a complete set of orthonormal spherical harmonics in p dimensions, with n denoting the order of the function. Then we know that there are $N(n,p)$ linearly independent spherical harmonics of order n, where $N(n,p)$ is given by

(2.1) $$\frac{1 + x}{(1 - x)^{p-1}} = \sum_{n=0}^{\infty} N(n,p)\, x^n .$$

As these spherical harmonics satisfy

(2.2) $$\Delta |x|^n \; S_{n,j}(\xi) = 0$$

we have

(2.3) $$\Delta |x|^n S_{n,j}(\xi) = n(n + p - 2)\, |x|^{n-2} S_{n,j} + |x|^n \; \Delta S_{n,j} ,$$

that is,

(2.4) $$\Delta S_{n,j} = - \frac{n(n + p - 2)}{|x|^2} \; S_{n,j} .$$

We introduce spherical coordinates by means of the equation

(2.5) $$(x) = r(\xi)$$

and find that the Laplacian may be written as

(2.6) $$\Delta = \frac{1}{r^{p-1}} \frac{\partial}{\partial r} \, r^{p-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \; \Delta_o$$

where Δ_o is the second Beltrami operator on the unit sphere in p dimensions. From (4) we have therefore

(2.7) $$\Delta_o \; S_{n,j}(\xi) = - n(n + p - 2)\, S_{n,j}(\xi).$$

Now we write

(2.8) $$F_{n,j}(R) = \int_{\Omega_1} U(R\xi)d\omega$$

where $d\omega$ is the surface element on the unit sphere. For $R \neq 1$ this function is twice differentiable and we have from (6)

(2.9)
$$\frac{1}{R^{p-1}} (F_{n,j}{}'(R)\, R^{p-1})'$$
$$= \int_{\Omega_1} S_{n,j}(\xi) \frac{1}{R^{p-1}} \frac{d}{dR} R^{p-1} \frac{d}{dR} U(R\xi)\, d\omega$$
$$= -\frac{1}{R^2} \int_{\Omega_1} S_{n,j}(\xi)\, \Delta_0\, U(R\xi)\, d\omega + \int_{\Omega_1} S_{n,j}(\xi)\, \Delta U\, d\omega$$
$$= -\frac{1}{R^2} \int_{\Omega_1} U(R\xi)\, \Delta_0\, S_{n,j}(\xi)\, d\omega + \int_{\Omega_1} S_{n,j}(\xi)\, \Delta U\, d\omega .$$

Introducing the relation

(2.10)
$$\phi_{n,j}(R) = \int_{\Omega_1} S_{n,j}(\xi)\, \Delta U\, d\omega$$

we find from (7) that

(2.11)
$$\frac{1}{R^{p-1}} (F_{n,j}{}'\, R^{p-1})' - \frac{n(n+p-2)}{R^2} F_{n,j} = \phi_{n,j} .$$

The general solution of this differential equation may be written as

(2.12)
$$F_{n,j}(R) = AR^n + \frac{B}{R^{n+p-2}} + \frac{R^n}{2n+p-2} \int_0^R \left(\frac{1}{r^{n-1}} - \frac{r^{n+p-1}}{R^{2n+p-2}}\right) \phi_{n,j}(r)\, dr.$$

Since $F_{n,j}(R) \varepsilon C'$ this holds for all R. Because of our assumptions, $\phi_{n,j}(r)$ vanishes more strongly than any power of r. Therefore the integral in (12) vanishes more strongly than any power, and since $F_{n,j}(R)$ itself is of the order of $o(R^n)$ we have

(2.13)
$$F_{n,j}(R) = \frac{R^n}{(2n+p-2)} \int_0^R \left(\frac{1}{r^{n-1}} - \frac{r^{n+p-1}}{R^{2n+p-2}}\right) \phi_{n,j}(r)\, dr.$$

On the other hand we have $\phi_{n,j}(r) = O(r^\beta)$ for $r \to \infty$ and we find that for $n > \beta + 2$

$$(2.14) \qquad AR^n + \frac{B}{R^{n+p-2}} - \frac{R^n}{2n+p-2} \left(\int_R^\infty \frac{\phi_{n,j}(r)}{r^{n-1}} dr + \int_0^R \frac{r^{n+p-1}\phi_{n,j}(r)dr}{R^{2n+p-2}} \right)$$

is also a general solution of (11). Since $F_{n,j}(R) = o(R^n)$ for $R \to 0$, we have $B = 0$, and since $F_{n,j}(R) = O(R^\beta)$ for $R \to \infty$ we have $A = 0$. Therefore, besides (13), we get

$$(2.15) \qquad F_{n,j}(R) = -\frac{R^n}{(2n+p-2)} \left[\int_R^\infty \frac{\phi_{n,j}(r)}{r^{n-1}} dr + \int_0^R \frac{r^{n+p-1}\phi_{n,j}(r)}{R^{2n+p-2}} dr \right],$$

for $n > \beta + 2$. Since

$$(2.16) \qquad \int_{\Omega_R} | \Delta U|^2 \, ds \leq c \int_{\Omega_R} |U|^2 \, ds,$$

and since the spherical harmonics are complete, we have

$$(2.17) \qquad \sum_{n=0}^{\infty} \sum_{j=1}^{N(p,n)} (\phi_{n,j}(R))^2 \leq c \sum_{n=0}^{\infty} \sum_{j=0}^{N(p,n)} |F_{n,j}(R)|^2.$$

Assume now that when $2\ell \geq \beta$,

$$(2.18) \qquad \int_{\Omega_1} |U(R\xi)|^2 \, d\omega \leq AR^{4\ell+1/2}.$$

Then

$$(2.19) \qquad \phi^2_{n,j}(R) = O(R^{4\ell+1/2}),$$

and for $n \leq 2(\ell+1)$ we have from (13)

$$(2.20) \qquad |F_{n,j}(R)|^2 \leq \frac{R^{2n}}{(2n+p-2)^2} \int_0^R \frac{|\phi_{n,j}|^2\, dr}{r^{4\ell+(1/2)+(3/4)}} \int_0^R r^{4\ell+(5/4)} \times$$

$$\times \left(\frac{1}{r^{n-1}} - \frac{r^{n+p-1}}{R^{2n+p-2}} \right)^2 dr = \frac{R^{4(\ell+1)+(1/4)}}{(2n+p-2)^2} \int_0^R \frac{|\phi_{n,j}|^2\, dr}{r^{4+(5/4)}} \times$$

$$\times \left[\frac{1}{4\ell+(5/4)+p-1-(2n+p-2)} - \frac{2}{4\ell+(5/4)+p+1} + \frac{1}{4\ell+(5/4)+p+1+(2n+p-2)} \right]$$

$$= \frac{2R^{4(\ell+1)+(1/4)}}{(4\ell+(9/4)+p)\left[(4\ell+(9/4)+p)^2 - (2n+p-2)^2\right]} \int_0^R \frac{|\phi_{n,j}|^2 dr}{r^{4\ell+(5/4)}} .$$

Hence

$$(2.21) \qquad |F_{n,j}(R)|^2 \leq \frac{R^{4(\ell+1)+(1/4)}}{2\ell^2} \int_0^R \frac{|\phi_{n,j}|^2}{r^{4\ell+(5/4)}} .$$

This last inequality holds because the denominator in (21) is the product of the denominators of the fractions in (20) and thus is greater than $4\ell^2$.

For $n > 2(\ell+1) \geq \beta + 2$ we find from (15)

$$(2.22) \quad |F_{n,j}(R)|^2 \leq \frac{2R^{2n}}{(2n+p-2)^2} \left[\int_R^\infty \frac{\phi_{n,j} dr}{r^{n-1}} \;{}^2 + \left(\int_0^R \frac{r^{n+p-1}\phi_{n,j}(r) dr}{R^{2n+p-2}} \right)^2 \right]$$

$$\leq \frac{2R^{2n}}{(2n+p-2)^2} \left[\int_R^\infty \frac{|\phi_{n,j}|^2 dr}{r^{4\ell+(5/4)+(1/2)}} \int_0^R r^{4\ell+(5/4)+(1/2)-2n+2}\, dr + \right.$$

$$\left. + \int_0^R \frac{|\phi_{n,j}|^2 dr}{r^{4\ell+(3/4)+(1/2)}} \int_0^R \frac{r^{4\ell+(5/4)+2n+2p-2}}{R^{4n+2p-4}}\, dr \right]$$

$$\leq \frac{2R^{4(\ell+1)+(3/4)}}{(2n+p-2)^2} \left[\frac{-1}{4(\ell+1)+(3/4)-2n} \int_R^\infty \frac{|\phi_{n,j}|^2 dr}{r^{4\ell+(5/4)+(1/2)}} + \right.$$

$$\left. + \frac{R^{-1/2}}{4\ell+(5/4)+2n+2p-1} \int_0^R \frac{|\phi_{n,j}|^2\, dr}{r^{4\ell+(3/4)+(1/2)}} \right] .$$

Because $n > 2(\ell+1)$ and $p \geq 2$ we have

$$|F_{n,j}(R)|^2 \leq \frac{R^{4(\ell+1)+(3/4)}}{2\ell^2} \int_R^\infty \frac{|\phi_{n,j}|^2\,dr}{r^{4\ell+(5/4)+(1/2)}} + \frac{R^{4(\ell+1)+(1/4)}}{2\ell^2} \int_0^R \frac{|\phi_{n,j}|^2\,dr}{r^{4\ell+(3/4)+(1/2)}} \; . \tag{2.23}$$

Consequently we get from (20) and (23) for $2\ell > \beta$ and any integer M

$$\sum_{n=0}^{M} \sum_{j=1}^{N(n,p)} |F_{n,j}(R)|^2 \leq \frac{R^{4(\ell+1)+(1/4)}}{2\ell^2} \int_0^R \frac{1}{r^{4\ell+(5/4)}} \sum_{n=0}^{M} \sum_{j=1}^{N(n)} |\phi_{n,j}(r)|^2\,dr \tag{2.24}$$

$$+ \frac{R^{4(\ell+1)+(3/4)}}{2\ell^2} \int_R^\infty \frac{1}{r^{4\ell+(1/2)+(5/4)}} \sum_{n=2(\ell+1)+1}^{M} \sum_{j=1}^{N(n)} |\phi_{n,j}(r)|^2\,dr.$$

Now, because

$$\sum_{n=0}^{\infty} \sum_{j=1}^{N} |\phi_{n,j}|^2 \leq c \sum_{n=0}^{\infty} \sum_{j=1}^{N} |F_{n,j}(R)|^2 \leq AcR^{4\ell+(1/2)} \quad , \tag{2.25}$$

we have from (24)

$$\sum_{n=0}^{\infty} \sum_{j=1}^{N(n)} |F_{n,j}(R)|^2 \leq \frac{Ac}{2\ell^2} R^{4(\ell+1)+(1/4)} \int_0^R \frac{dr}{r^{(3/4)}} \tag{2.26}$$

$$+ \frac{Ac}{2\ell^2} R^{4(\ell+1)+(3/4)} \int_R^\infty \frac{dr}{r^{(5/4)}}$$

$$= \frac{4Ac}{2} R^{4(\ell+1)+(1/2)} \quad .$$

From the fact that

$$(2.27) \qquad \int_{\Omega_1} |U(R\xi)|^2 \, d\omega \leq AR^{4\ell+(1/2)}$$

it follows that under the assumptions of Lemma 1

$$(2.28) \qquad \int_{\Omega_1} |U(R\xi)|^2 \, d\omega \leq \frac{4Ac}{\ell^2} \, R^{4(\ell+1)+(1/2)} .$$

Now the inequality (27) certainly holds for $2\ell > \beta$. For $L = [\frac{\beta}{2}] + 1$ we finally get

$$(2.29) \qquad \int |U(R\xi)|^2 \, d\omega \leq AR^{4L+(1/2)} \frac{R^n}{L(L+1)\dots(L+n-1)}$$

for all n = 1, 2, ... ; the right-hand side tends towards zero as $n \to \infty$. This completes the proof of Lemma 1.

III. Proof of Lemma 2

In order to prove Lemma 2 we have to construct a function $U(x) \varepsilon C''$ for $|x| > 1$ such that

$$(3.1) \qquad \lim_{R \to 1+0} \int_{\Omega_1} U(R\xi) \; S_n(\xi) ds = \int_{\Omega_1} f(\xi) S_n(\xi) ds$$

$$\lim_{R \to 1+0} \frac{d}{dR} \int_{\Omega_1} U(R\xi) \; S_n(\xi) ds = \int_{\Omega_1} g(\xi) \; S_n(\xi) ds$$

for any spherical harmonic $S_n(\xi)$. Since in Lemma 2 we have assumed $f(\xi) \varepsilon C''$ and $g(\xi) \varepsilon C'$, we write

(3.2) $$U(x) = f\left(\frac{x}{|x|}\right) + \sum_{n=0}^{\infty} \left(\frac{1}{|x|^{n+p-1}} - \frac{1}{|x|^{n+p}}\right) S_n(\xi),$$

where

(3.3) $$S_n(\xi) = \sum_{j=1}^{N(p,n)} C_{n,j}\, S_{n,j}(\xi)$$

with

(3.4) $$C_{n,j} = \int_{\Omega_1} g(\xi)\; S_{n,j}(\xi)\, d\omega \;.$$

This function $U(x)$ belongs to C'' for $|x| > 1$, and we have

(3.5) $$\lim_{R \to 1} \int_{\Omega_1} |U(R\xi)|^2\, d\omega = \int_{\Omega_1} |f(\xi)|^2\, d\omega > 0$$

$$\lim_{R \to \infty} \int_{\Omega_1} |U(R\xi)|^2\, d\omega = \int_{\Omega_1} |f(\xi)|^2\, d\omega > 0.$$

Now put

(3.6) $$C'_{n,j} = \int_{\Omega_1} f(\xi)\; S_{n,j}(\xi)\, d\omega \;,$$

and suppose

(3.7) $$\int_{\Omega_1} |U(R_0\xi)|^2\, d\omega = 0$$

for some $R_0 \to 1$. Then it follows from (4) that

$$(3.8) \qquad C'_{n,j} = - \frac{1}{R_o^{n+p-1}} \left(1 - \frac{1}{R_o}\right) C_{n,j}$$

holds for all n. Therefore (8) can be satisfied for at most one value of R_o.

First let us assume that no such R_o exists. Then because of (5) there is a positive constant b such that

$$(3.9) \qquad \int_{\Omega_1} |U(R\xi)|^2 \, d\omega \geq b > 0,$$

for all $R \geq 1$.

Let ∇_o and Δ_o be, respectively, the first and second Beltrami operators on the unit sphere in p dimensions. Since $g(\xi) \varepsilon C'$, we have

$$(3.10) \qquad 0 \leq \int_{\Omega_1} \left|\nabla_o f(\xi) - \sum_{n=0}^{M} \nabla_o S_n(\xi)\right|^2 d\omega$$

$$= \int_{\Omega_1} |\nabla_o f|^2 \, d\omega - 2 \sum_{n=0}^{M} \int_{\Omega_1} (\nabla_o f \cdot \nabla_o S_n) d\omega + \sum_{n=0}^{M} \int_{\Omega_1} |\nabla_o S_n|^2 \, d\omega,$$

$$(3.11) \qquad \int_{\Omega_1} (\nabla_o S_n \cdot \nabla_o S_m) \, d\omega = - \int_{\Omega_1} S_n \, \Delta_o \, S_m d\omega = n(n + p - 2)\delta_{nm} \int_{\Omega_1} |S_n|^2 \, d\omega ,$$

Put

$$(3.12) \qquad c_n^2 = \int_{\Omega_1} f(\xi) \, S_n(\xi) \, d\omega = \sum_{j=1}^{N} |C_{n,j}|^2 = \int_{\Omega_1} |S_n(\xi)|^2 \, d\omega \ .$$

Then

$$(3.13)\qquad \int_{\Omega_1} f(\xi)\, S_n(\xi)\, d\omega = C_n^2 ,$$

and we find

$$(3.14)\qquad \int_{\Omega_1} (\nabla_o f \cdot \nabla_o S_n) d\omega = -\int_{\Omega_1} f\, \Delta_o\, S_n\, d\omega = n(n + p - 2)\, C_n^2 ,$$

so that from (10) we get

$$(3.15)\qquad \sum_{n=0}^{\infty} n^2\, C_n^2 < \infty .$$

It follows from (3) that for $|x| > 1$.

$$(3.16)\qquad \Delta\, U(x) = \Delta\, f(\xi) + \Delta \sum_{n=0}^{\infty} \left(\frac{1}{|x|^{n+p-1}} - \frac{1}{|x|^{n+p}} \right) S_n(\xi)$$

$$= \frac{1}{|x|^2}\, \Delta_o\, f(\xi) + \frac{1}{|x|^2} \sum_{n=0}^{\infty} \left(\frac{2n + p - 1}{|x|^{n+p-1}} - \frac{2(2n + p)}{|x|^{n+p}} \right) S_n(\xi) .$$

It is seen that the sum on the right-hand side of (16) represents a square integrable function of (ξ) for any (x), $|x| \geq 1$, and hence we have

$$(3.17)\qquad \lim_{R \to \infty} \frac{1}{R^2} \int_{\Omega_1} |\Delta\, U|^2\, d\omega = 0.$$

Therefore there is a positive constant c such that

$$(3.18) \qquad \frac{\displaystyle\int_{\Omega_R} |\Delta U|^2 \, ds}{\displaystyle\int_{\Omega_R} |U|^2 \, ds} \leq c .$$

However, if (8) holds for a certain R_o and for all n, we write instead of (2)

$$(3.19) \qquad U(x) = f\left(\frac{x}{|x|}\right) + \sum_{n=0}^{\infty} \left(\frac{1}{|x|^{n+p}} - \frac{1}{|x|^{n+p+1}} \right) S_n(\xi) .$$

This function again belongs to C'' for $|x| > 1$ and satisfies the limit relations (5). We have for $R > 1$

$$(3.20) \qquad \int_{\Omega_1} |U(R\xi)|^2 \, d\omega = \int_{\Omega_1} \left[f(\xi) - \sum_{n=0}^{\infty} \left(\frac{1}{R^{n+p}} - \frac{1}{R^{n+p-1}} \right) S_n(\xi) \right]^2 d\omega,$$

and this vanishes for some R_1 only if

$$(3.21) \qquad C'_{n,j} = - \frac{1}{R_1^{n+p}} \left(1 - \frac{1}{R_1}\right) C_{n,j}$$

for all n. If $f(\xi)$ is not equal to a spherical harmonic of order n, it follows from (21) and (8) that

$$(3.22) \qquad \frac{1}{R_o^{n+p-1}} \left(1 - \frac{1}{R_o}\right) = \frac{1}{R_1^{n+p}} \left(1 - \frac{1}{R_1}\right)$$

for at least two values of n. Since this is equivalent to

$$(3.23) \qquad \left(\frac{R_1}{R_o}\right)^{n+p} = \frac{1}{R_1} \cdot \frac{R_1 - 1}{R_o - 1} \quad ,$$

we see that (22) can hold for more than one value of n only if $R_1 = R_o = 1$. But this is impossible because of (8).

Therefore either $f(\xi)$ is a spherical harmonic of order n and $g(\xi) = - a f(\xi)$, where $0 < a < 1$, or at least one of the expansions

$$(3.24) \qquad \int_{\Omega_1} \left[f(\xi) + \sum_{n=0}^{\infty} \left(\frac{1}{R^{n+p-1}} - \frac{1}{R^{n+p}} \right) S_n(\xi) \right]^2 d\omega$$

$$\int_{\Omega_1} \left[f(\xi) + \sum_{n=0}^{\infty} \left(\frac{1}{R^{n+p}} - \frac{1}{R^{n+p+1}} \right) S_n(\xi) \right]^2 d\omega$$

is always positive for $R \geq 1$.

Using (20) we find the following relation corresponding to (16):

$$(3.25) \qquad \triangle U = \frac{1}{|x|^2} \triangle_o f(\xi) + \frac{1}{|x|^2} \sum_{n=0}^{\infty} \left(\frac{2(2n+p)}{|x|^{n+p}} - \frac{3(2n+p+1)}{|x|^{n+p+1}} \right) S_n(\xi) \; .$$

Here again we see that for all R

$$(3.26) \qquad \frac{1}{R^{p-1}} \int_{\Omega_R} \left| \triangle U \right|^2 ds = O(1),$$

and hence

$$\frac{\int_{\Omega_R} |\Delta U|^2 \, ds}{\int_{\Omega_R} |U|^2 \, ds} \leq c. \tag{3.27}$$

We have thus found an extension of our function which satisfies the requirements of Lemma 2 unless $f(\xi) = - a\, g(\xi)$ with $0 < a < 1$. In this case we put

$$U(x) = U(r\xi) = f(\xi)(1 - a + \frac{a}{r}) \tag{3.28}$$

and find that

$$\int_{\Omega_1} |U(R\xi)|^2 \, d\omega = \left(1 - a + \frac{a}{R}\right)^2 \int_{\Omega_1} |f(\xi)|^2 \, d\omega > 0 \tag{3.29}$$

for $R \geq 1$, while the integral

$$R^{1-p} \int_{\Omega_R} |\Delta U|^2 \, d\omega \tag{3.30}$$

is bounded again for $R \geq 1$.

This completes the proof of Lemma 2.

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